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ON THE DISCONTINUITY OF THE FLUTTER LOAD FOR VARIOUS TYPES OF CANTILEVERS

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Abstract—In this investigation, using an energy (variational) approach, the flutter instability for various types of elastically restrained uniform cantilevers carrying up to three concentrated masses and subjected to a follower compressive force, is presented. The effects of transverse shear deformation and rotatory inertia of the mass of the column and of the positioning of the concentrated masses with or without their rotational inertia, are also included in the analysis.

In all cases, where the flutter load is obtained from the coincidence of the second and third flexural eigenfrequencies a discontinuity with a finite jump in this load is possible; the lower value of the flutter load at this discontinuity is obtained from the coincidence of the first and second eigenfrequencies, while the upper value of this load is obtained from the coincidence of the second and third eigenfrequencies. Subsequently, it is found that the flutter load is a sectionally continuous function of certain of the varying parameters. Finally, the effect of several parameters upon the magnitude of the jump in the flutter load, is also discussed.

INTRODUCTION

Recently, a considerable amount of work has been published on the problem of elastic stability of structural systems under the action of nonconservative follower type forces which are of particular importance in modern engineering practice. Such systems according to Leipholz' [1] classification may be nonconservative of the second kind, if their loss of stability occurs through divergence, or pure nonconservative, if their loss of stability occurs through flutter. Herrmann and Bungay [2] have shown that the divergence or flutter type instability of the aforementioned systems depends on the geometric boundary conditions. A further contribution to this problem has been presented recently by Kounadis [3]; in this investigation it has also been proven that an elastically restrained (or fully fixed) cantilever subjected to a follower compressive force at its free end loses always its stability through flutter. Accordingly the corresponding critical (flutter) load can be established by using only the dynamic method. Kounadis and Katsikadelis [4, 5] have investigated the effect of shear deformation and rotatory inertia on the flutter instability of an elastically restrained Beck's column carrying concentrated masses at one or both ends. Also Kounadis [6], in a more general analysis, has studied the individual and coupling effects of the foregoing parameters including the influence of the positioning along the length of the column of several concentrated masses with or without rotational inertia.

In Refs. [4–6] was also observed that the flutter load may be derived from the coalescing of the second and third flexural eigenfrequencies; moreover, in studying the effect of various parameters upon the flutter load the following phenomenon was observed. In some cases where the flutter load is obtained by the coalescing of the first and second eigenfrequencies, after a further increase of the load beyond its critical value, the first and second eigenfrequencies reappear. This usually occurs for a small range of values of the varying parameter at the end of which the flutter load is obtained by the coincidence of the second and third eigenfrequencies.

The objective of the present investigation based on the kinetic criterion, is to discuss and clarify the aforementioned phenomenon. To this end the flutter instability of various types of elastically restrained cantilevers of uniform cross section carrying up to three concentrated masses and subjected to a follower compressive force of constant magnitude at its tip is investigated. The effects of transverse shear and rotatory inertia of the mass of the column, of the positioning of the concentrated masses as well as of their rotational inertia are also included in this analysis.

The Timoshenko's shear coefficient is evaluated by using Cowper's [7] formulae. The governing equations of motion and associated boundary conditions are established by using an energy (variational) approach; this is an extension of Hamilton's stationary principle to nonconservative problems.

The problem is reduced to a system of two (linear) coupled partial differential equations involving generalized functions due to the presence of a concentrated mass. Subsequently a closed form solution is succesfully obtained by using Laplace transforms.

MATHEMATICAL ANALYSIS

Consider the uniform cantilever of length l, flexural rigidity EI, cross-sectional area A and material density ρ, shown in Fig. 1. The cantilever is supported on an elastic rotational spring with stiffness C_R and on an elastic translational spring with stiffness C_T, it carries three concentrated masses M_1, M_2, M_3 with respective rotational inertias J_1, J_2, J_3, and it is acted upon by a follower compressive force S, applied at its tip. Throughout this analysis it is assumed that the masses M_1, M_3 are attached at the elastic support and the free end of the cantilever, respectively, while the mass M_2 is located at an arbitrary distance α from the elastic support. For the sake of a more accurate[6] investigation the effects of transverse shear deformation and rotatory inertia of the cantilever are also included in this analysis.

According to the kinetic criterion sufficiently small lateral displacements from the equilibrium position are imposed and the resulting disturbed motion about this position is discussed. Let $y(x, t)$ be the total lateral deflection and $\psi(x, t)$ the angle of rotation of the cross section (due only to bending), where x is the axial coordinate and t the time. For the bending moment $M(x, t)$ and the shearing force $Q(x, t)$ normal to the deformed axis of the cantilever, the following expressions are valid

$$M(x, t) = -EI\psi'(x, t)$$
$$Q(x, t) = k'AG[y'(x, t) - \psi(x, t)] \tag{1}$$

where k' is Timoshenko's shear coefficient evaluated by means of Cowper's formulae [7] and G is the shear modulus; the prime denotes differentiation with respect to x.

Within the scope of the linear stability theory the follower force S may be resolved into a vertical (conservative) component $S\cos y'(l, t) \simeq S$ and a horizontal (nonconservative) component $S\sin y'(l, t) \simeq Sy'(l, t)$.

Application of the stationary (value) principle of Hamilton for the foregoing nonconservative system, as indicated by Levinson [8], implies

$$\delta^{(1)}\int_{t_1}^{t_2}(T - U^T)\,dt = 0, \quad \delta^{(1)}[Sy'(l, t)] = 0 \tag{2}$$

where the functionals of the kinetic energy and total potential energy are given by

$$T = \frac{1}{2}\int_0^l [\rho A\dot{y}^2 + \rho I\dot{\psi}^2]\,dx + \frac{1}{2}[M_1\dot{y}^2(0, t) + J_1\dot{\psi}^2(0, t)$$
$$+ M_2\dot{y}^2(\alpha, t) + J_2\dot{\psi}^2(\alpha, t) + M_3\dot{y}^2(l, t) + J_3\dot{\psi}^2(l, t)];$$

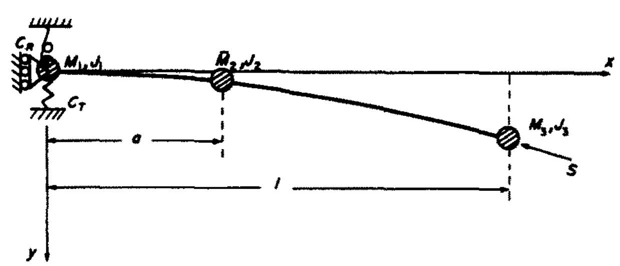

Fig. 1. Elastically restrained cantilever carrying three concentrated masses M_1, M_2, M_3 and subjected to a follower compressive force S.

$$U^T = \frac{1}{2}\int_0^l [EI\psi'^2 + k'AG(y' - \psi)^2]\,\mathrm{d}x + \frac{1}{2}C_T y^2(0,t) + \frac{1}{2}C_R\psi^2(0,t)$$
$$-\frac{1}{2}\int_0^l Sy'^2\,\mathrm{d}x + Sy'(l,t)y(l,t) \tag{3}$$

$\delta^{(1)}$ is the first variation; the dot denotes differentiation with respect to time t.

The aforegoing Hamilton's principle by means of relation (3) and taking into account that

$$M_2\dot{y}^2(\alpha,t) + J_2\dot{\psi}^2(\alpha,t) = \int_0^l \delta(x-\alpha)[M_2\dot{y}^2 + J_2\dot{\psi}^2]\,\mathrm{d}x \tag{4}$$

yields the following variational equations

$$[\rho I + J_2\delta(x-\alpha)]\ddot{\psi} - k'AG(y' - \psi) - EI\psi'' = 0$$
$$[\rho A + M_2\delta(x-\alpha)]\ddot{y} - k'AG(y'' - \psi') + Sy'' = 0 \tag{5}$$

where δ denotes the generalized Dirac function. The resulting boundary conditions are at $x = 0$

$$EI\psi'(0,t) = C_R\psi(0,t) + J_1\ddot{\psi}(0,t)$$
$$k'AG[y'(0,t) - \psi(0,t)] = C_T y(0,t) + Sy'(0,t) + M_1\ddot{y}(0,t)$$

at $x = l$

$$EI\psi'(l,t) = -J_3\ddot{\psi}(l,t)$$
$$k'AG[y'(l,t) - \psi(l,t)] = -M_3\ddot{y}(l,t) \tag{6}$$

Setting $M_1 = J_1 = 0$ eqns (5) and (6) are in agreement with eqns (3) and (4) or Ref. [6].

Equations (5) and (6) describe the perturbed motion around the original position which is stable (in the sense of Lyapunov's definition), if the lateral displacements $y(x,t)$, $\psi(x,t)$ remain for all times sufficiently small for sufficiently small initial perturbations. These variational equations determine whether or not this is possible. The initial conditions which have led to the foregoing displacements are not taken into account. This is so, for it is not the transient response but the steady-state response which determines the stability or instability of the system; apparently this response depends on the differential eqns (5) and the corresponding boundary conditions (6). Hence, one may assume solutions corresponding to the steady-state response of the form

$$y(x,t) = \overset{*}{y}(x)\,\mathrm{e}^{i\omega t}$$
$$\psi(x,t) = \overset{*}{\psi}(x)\,\mathrm{e}^{i\omega t} \tag{7}$$

in which "i" is the imaginary unit; ω is the circular frequency of the steady-state motion and $\overset{*}{y}(x)$, $\overset{*}{\psi}(x)$ are the eigenmodes of this vibratory motion.

Note that the modal analysis, on which the present results depend, presupposes that the corresponding eigenmodes are complete. If this is so it has not yet been proven to the knowledge of the authors.

Introduction of the nondimensionalized quantities

$$\xi = \frac{x}{l};\ \bar{\alpha} = \frac{\alpha}{l};\ Y(\xi) = \frac{\overset{*}{y}(x)}{l},\ \Psi(\xi) = \overset{*}{\psi}(X)$$
$$\Omega^2 = \frac{\omega^2\rho A l^4}{EI};\ \lambda^2 = \frac{Al^2}{I};\ r^2 = \lambda^2 k'\frac{G}{E}$$

$$\bar{s}=\frac{Sl^2}{EI};\ s=1-\frac{S}{k'AG};\ \frac{S}{k'AG}=\frac{\bar{S}}{\lambda^2k'};\ \frac{E}{G};\ \frac{E}{G}=2(1+\nu);$$

$$\bar{M}_i=\frac{M_i}{l\rho A};\ \bar{J}_i=\frac{J_i}{l^3\rho A};\ \bar{C}_R=\frac{C_Rl}{EI};\ \bar{C}_T=\frac{C_Tl^3}{EI};\qquad i=1,2,3;$$

and substitution of relations (7) into eqns (5) and (6) imply

$$\left.\begin{aligned} s\{Y''(\xi)+k_1^4[1+\bar{M}_2\delta(\xi-\bar{\alpha})]\}Y(\xi)-\Psi'(\xi)=0\\ \{sk_2^4[\lambda^{-2}+\bar{J}_2\delta(\xi-\bar{\alpha})]-r^2\}\Psi(\xi)+\Psi''(\xi)+r^2Y'(\xi)=0\end{aligned}\right\} \tag{8}$$

and

$$\left.\begin{aligned} \text{at}\quad \xi=0:\quad & \Psi'(0)=\bar{C}_R\Psi(0);\ sY'(0)=\Psi(0)+\frac{\bar{C}_T}{r^2}Y(0)\\ \text{at}\quad \xi=1:\quad & \Psi'(1)=sk_2^4\bar{J}_3\psi(1);\ Y'(1)=\Psi(1)+sk_1^4\bar{M}_3Y(1)\end{aligned}\right\} \tag{9}$$

where

$$k_1^4=\frac{\Omega^2}{sr^2};\quad k_2^4=\frac{\Omega^2}{s}. \tag{10}$$

Subsequently, following the same solution procedure given in Ref. [6] it follows that

$$\begin{aligned} Y(\xi)=\varphi_1(\xi)Y(0)+\varphi_2(\xi)Y'(0)+H(\xi-\bar{\alpha})[\bar{M}_2Y(\bar{\alpha})(k^4F_2(\xi-\bar{\alpha})-k_1^4F_2''(\xi-\bar{\alpha}))\\ -k_2^4\bar{J}_2\Psi(\bar{\alpha})F_2'(\xi-\bar{\alpha})]\end{aligned}$$

$$\begin{aligned}\Psi(\xi)=\varphi_3(\xi)Y(0)+\varphi_4(\xi)Y'(0)+k_2^4H(\xi-\bar{\alpha})[\bar{M}_2Y(\bar{\alpha})F_2'(\xi-\bar{\alpha})\\ -s\bar{J}_2\Psi(\bar{\alpha})(k_1^4F_2(\xi-\bar{\alpha})+F_2''p(\xi-\bar{\alpha})]\end{aligned} \tag{11}$$

where the functions $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\varphi_3(\xi)$, $\varphi_4(\xi)$ are given [6] in the Appendix; H is the Heaviside function; k^4 is given by

$$k^4=-\frac{\Omega^2}{s}\left(\frac{\Omega^2}{2\lambda^4k'(1+\nu)}-1\right). \tag{12}$$

Application of the boundary conditions (9) by virtue of eqns (11) leads to a homogeneous algebraic system of four equations with respect to $Y(0)$, $Y(\bar{\alpha})$, $\Psi(0)$, $\Psi(\bar{\alpha})$. For a nontrivial solution the determinant of this system must vanish yielding the frequency equation, which is a transcendental equation of the form

$$F(\Omega^2,\bar{s},k',\lambda^2,\nu,\bar{C}_R,\bar{C}_T,\bar{M}_i,\bar{J}_i,\bar{\alpha})=0,\qquad (i=1,2,3). \tag{13}$$

The nondimensionalized eigenfrequencies Ω^2 of the free flexural vibrations are established as functions of the nondimensionalized compressive follower force $\bar{s}$ and the dimensionless parameters k', λ^2, ν, $\bar{C}_R$, $\bar{C}_T$, $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$, $\bar{J}_1$, $\bar{J}_2$, $\bar{J}_3$, $\bar{\alpha}$. The critical loads are obtained by solving eqn (13) on a digital computer using a numerical scheme based on step increasing the magnitude of the follower compressive force, whereas the other parameters are kept constant. It should be noted that either for $\bar{M}_3\to\infty$ (implying $y(1)\to0$) or for $\bar{J}_3\to\infty$ (implying $y'(1)\to0$) the column loses its stability through divergence. Excluding the last two cases the column is associated with a flutter type instability (oscillations with increasing amplitude). The respective critical (flutter) load corresponds to that magnitude of the compressive force $\bar{s}$ for which the first coincidence of two consecutive eigenfrequencies occurs.

Next, four particular cases are considered in which the coupling effect of at least three parameters is discussed.

Case 1

This case corresponds to a partially fixed Timoshenko Beck's column carrying two concentrated masses, one placed at the support and the other at its tip. The Timoshenko's shear coefficient k' is equal to 0.186 and corresponds to a thin I-section with $t_f/d = 1/10$, $t_w/d = 1/20$, $b = h = d$, $\nu = 0.30$[7]. The respective frequency equation is established by setting into eqn (13) $\bar{C}_R = 1$, $\bar{C}_T \to \infty$, $\bar{M}_2 = \bar{J}_2 = 0$, $\bar{M}_3 = 0.5$, $\bar{J}_3 = 1$, $k' = 0.186$, $\lambda = 30$, $\nu \times 0.30$; $\bar{M}_1$ is arbitrary, while $\bar{J}_1$ is the varying parameter.

Case 2

This case corresponds to a partially fixed Timoshenko–Beck's column carrying only one mass placed at its support. Therefore, by setting $\bar{M}_3 = \bar{J}_3 = 0$ into the frequency equation of the previous case, one may obtain the corresponding equation of case 2.

Case 3

This case results from the preceding one by neglecting the effect of transverse shear deformation ($k' \to \infty$) and rotatory inertia.

Case 4

This case corresponds to a fully fixed Beck's column carrying a concentrated mass placed at a distance $\bar{\alpha}$. The respective frequency equation is derived from eqn (13) by setting $\bar{C}_T \to \infty$, $\bar{C}_R \to \infty$, $\bar{M}_2 = \bar{J}_2 = 0.5$, $k' \to \infty$, $\lambda^2 \to \infty$, $\bar{M}_3 = \bar{J}_3 = 0$.

NUMERICAL RESULTS AND DISCUSSION

Solving numerically the frequency equation corresponding to each of the aforementioned cases, one may determine the respective flutter load; this is established as function of the varying parameter $\bar{J}_1$ (cases 1–3) or of the varying position $\bar{\alpha}$ of a mass $\bar{M}_2 = \bar{J}_2 = 0.5$ (case 4). The obtained solutions are presented in a graphical form.

From Figs. 2(a–f) corresponding to case 1, one can see the relationship between the follower compressive load $\bar{s}$ and the first three flexural eigenfrequencies $\Omega_i (i = 1, 2, 3)$ for six different values of the rotational intertia $\bar{J}_1$ of an arbitrary mass $\bar{M}_1$ placed at the support of the cantilever. More specifically in Fig. 2(a) ($\bar{J}_1 = 1.90$) it is shown that the first and second flexural eigenfrequencies coalesce at point A to which corresponds the lowest flutter load $\bar{s}_F = 4.96$. For higher values of the load the two first flexural eigenfrequencies disappear. Moreover, it is observed that to each value of the load $\bar{s}$ corresponds only one point of the curve $\bar{s}$ vs Ω_3 (single-valued function). As the value of $\bar{J}_1$ increases to $\bar{J}_1 = 1.92$ (Fig. 2b) it is observed that the single valuedness of the function $\bar{s}$ vs Ω_3 is destroyed for a small (finite) range of $\bar{s}$ values between point B (local minimum) and C (local maximum). For a further increase in the $\bar{J}_1$ value to $\bar{J}_1 = 2.00$ (Fig. 2c) the foregoing range of $\bar{s}$ values between points B and C increases. Moreover the flutter load increases as $\bar{J}_1$ increases. It is also observed that as $\bar{J}_1$ increases to 2.00457 points A and B coincide (Fig. 2d). Furthermore, the first flutter load which increases with increasing $\bar{J}_1$ values, has attained its maximum value ($\bar{s}_F = 6.13$). Note that a second flutter load ($\bar{s}_F = 6.93$) corresponds to point C. It, thus, seems that between these two flutter loads there exists a post-critical stable region, because of the reappearence of the first two eigenfrequencies in the aforementioned region. From Fig. 2e, it is worth noticing that a further increase in the $\bar{J}_1$ value yields a flutter load $\bar{s}_F = 6.90$ derived from the coincidence of the second and third eigenfrequencies corresponding to point C. This phenomenon continues to exist for higher values of $\bar{J}_1$ with a continuous (slight) decrease of the flutter load (Fig. 2(f) and Fig. 3 for $\bar{J}_1 > 2.00457$). Thus, for $\bar{J}_1 = 2.00457^-$ the flutter load is equal to 6.13 while for $\bar{J}_1 = 2.00457^+$ the flutter load becomes equal to 6.93 and therefore the corresponding finite jump is equal to 0.80.

In conclusion on the basis of the aforegoing observations one may establish the effect of $\bar{J}_1$ upon the (first) flutter load (Fig. 3). Clearly from Figs. 2 and 3 one may see that: (a) For $\bar{J}_1 < 2.00457$ the flutter load is derived from the coincidence of the first and second eigenfrequencies, and it increases with increasing values of $\bar{J}_1$. (b) For $\bar{J}_1 > 2.00457$ the flutter load is derived from the coincidence of the second and third eigenfrequencies, and it decreases with increasing values of $\bar{J}_1$, approaching asymptotically the value of $\bar{s}_F = 4.69$ as $\bar{J}_1 \to \infty$. (c) for $\bar{J}_1 = 2.00457$ there is a discontinuity in the flutter load whose jump corresponds to an increase of

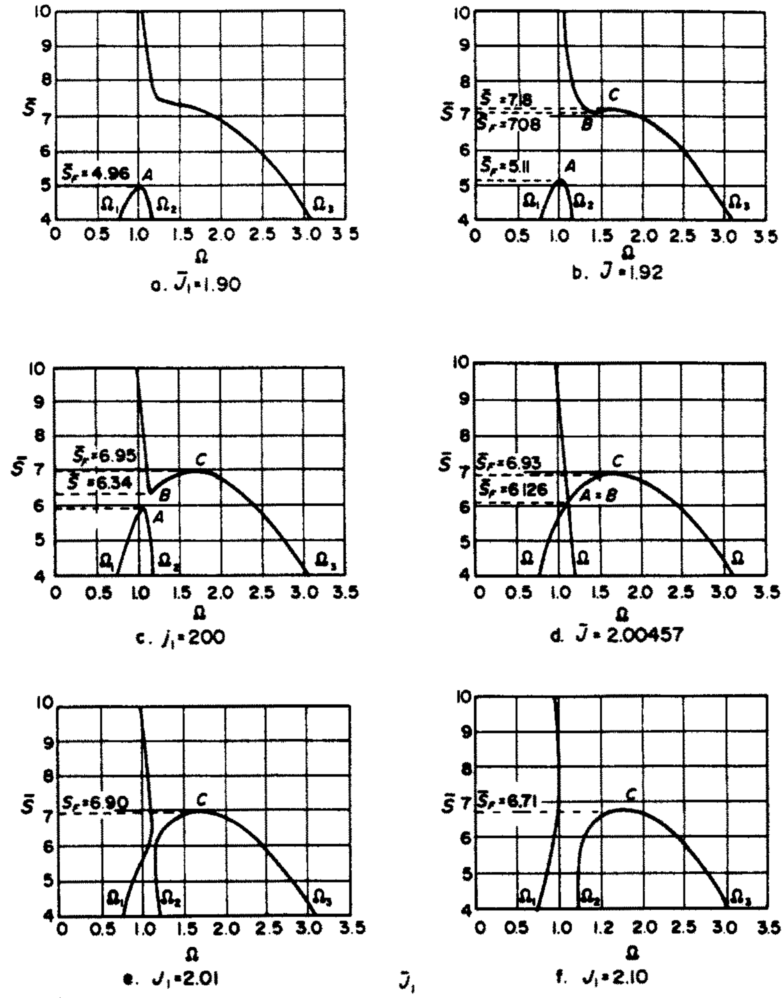

Fig. 2. Relationship between the dimensionless load $\bar{s}$ and the first three eigenfrequencies for a partially fixed ($\bar{C}_R = 1$) cantilever carrying two concentrated masses ($\bar{M}_1$ = arbitr., $\bar{J}_1$ = 1.90, 1.92, 2.00, 2.00457, 2.01, 2.10 and $\bar{M}_3 = 0.5$, $\bar{J}_3 = 1.0$).

the flutter load by about 13%. (d) There is only one jump in the flutter load function as $\bar{J}_1$ varies from zero to infinity. Consequently the flutter load is a sectionally (piece-wise) continuous function of $\bar{J}_1$.

A similar jump phenomenon in the flutter load function vs $\bar{J}_1$ has been also observed in case

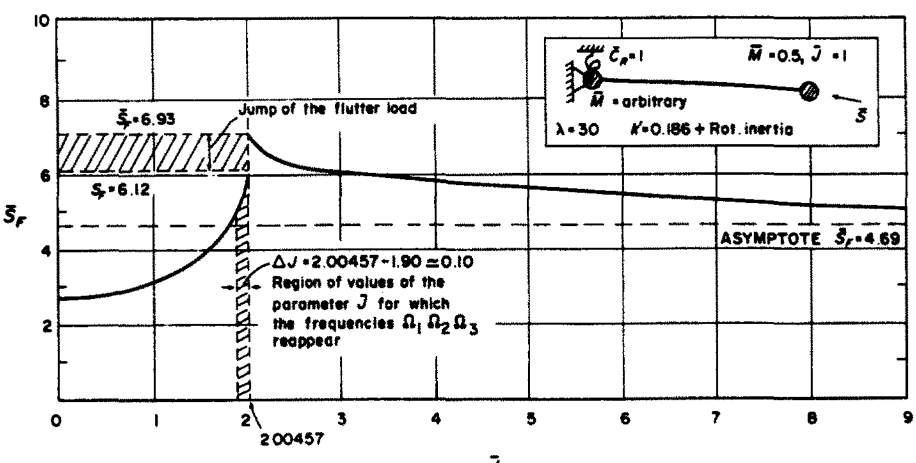

Fig. 3. Flutter load $\bar{s}_F$ vs moment of inertia $\bar{J}_1$ of an arbitrary concentrated mass $\bar{M}_1$ placed at the support of a partially fixed ($\bar{C}_R = 1$) cantilever carrying also a concentrated mass $\bar{M}_3$ at the tip.

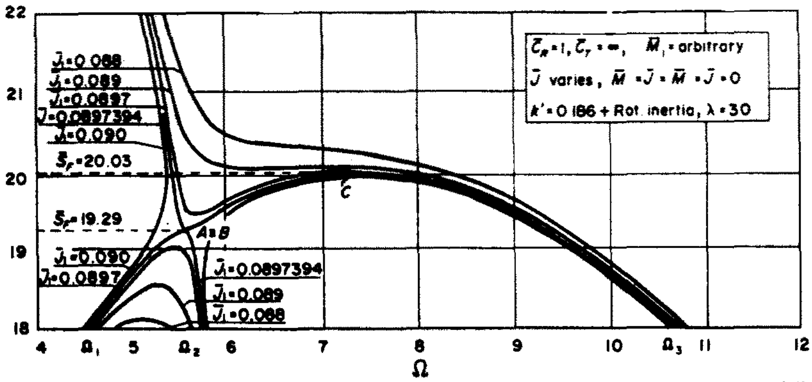

Fig. 4. Relationship between the dimensionless load $\bar{s}$ and the first three eigenfrequencies for a partially fixed ($\bar{C}_R = 1$) cantilever carrying an arbitrary concentrated mass $\bar{M}$ at the support with $J_1 = 0.088, 0.089, 0.0897, 0.0897394, 0.090$.

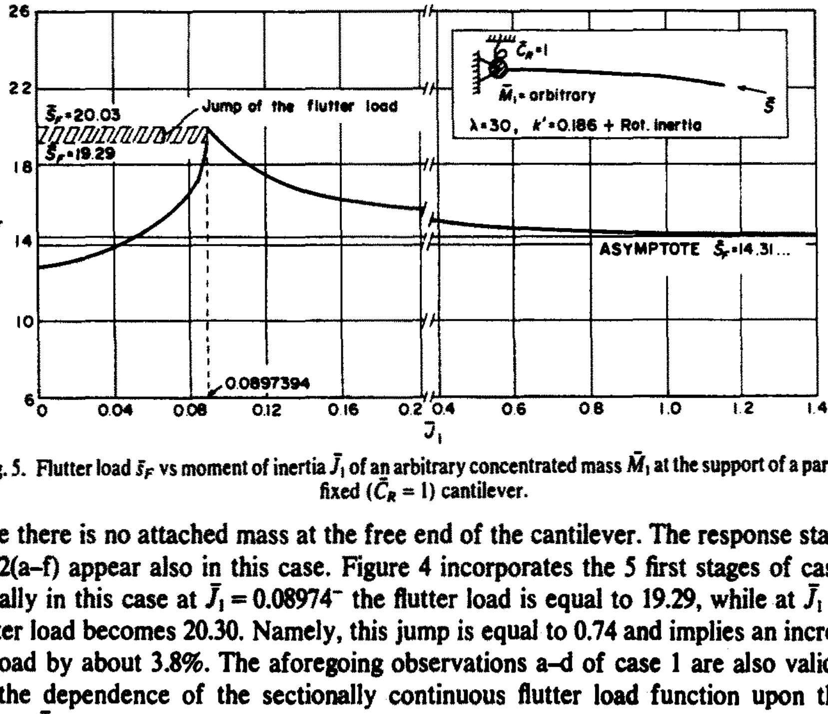

Fig. 5. Flutter load $\bar{s}_F$ vs moment of inertia $\bar{J}_1$ of an arbitrary concentrated mass $\bar{M}_1$ at the support of a partially fixed ($\bar{C}_R = 1$) cantilever.

2, where there is no attached mass at the free end of the cantilever. The response stages shown in Fig. 2(a–f) appear also in this case. Figure 4 incorporates the 5 first stages of case 2. More specifically in this case at $\bar{J}_1 = 0.08974^-$ the flutter load is equal to 19.29, while at $\bar{J}_1 = 0.08974^+$ the flutter load becomes 20.30. Namely, this jump is equal to 0.74 and implies an increase of the flutter load by about 3.8%. The aforegoing observations a–d of case 1 are also valid. Figure 5 shows the dependence of the sectionally continuous flutter load function upon the varying parameter $\bar{J}_1$. This Figure appears also in Ref. [4] but without the flutter jump details.

Similar results hold also for case 3, resulting from the previous case by neglecting the effect of transverse shear deformation and rotatory inertia. In this case, as shown in Fig. 6, the jump

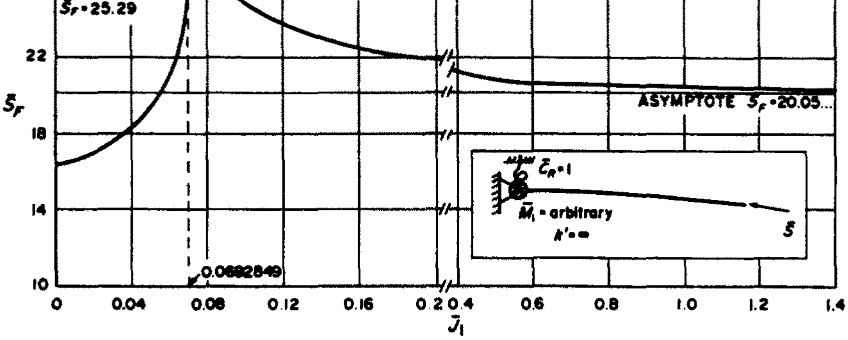

Fig. 6. Flutter load $\bar{s}_F$ vs moment of inertia $\bar{J}_1$ of an arbitrary concentrated mass $\bar{M}_1$ at the support of a partially ($\bar{C}_R = 1$) fixed cantilever neglecting the transverse shear and rotatory inertia effect.

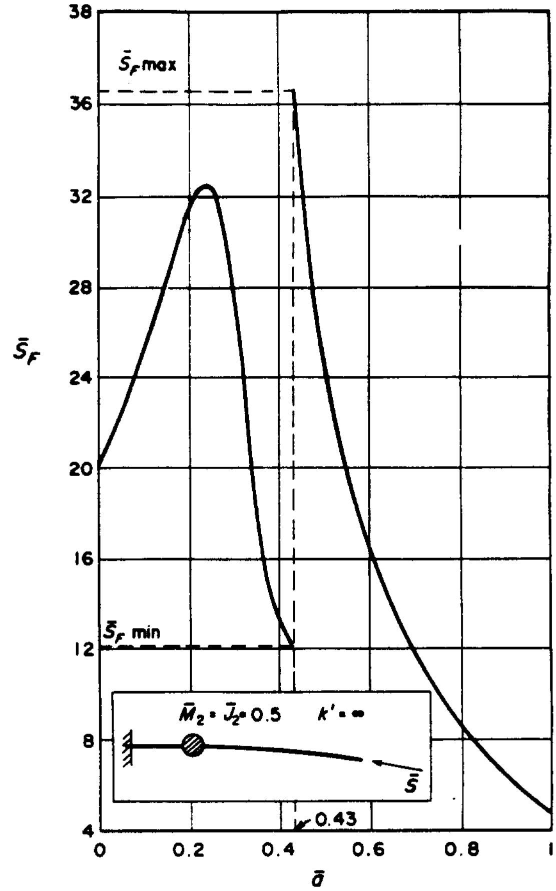

Fig. 7. Flutter load $\bar{s}_F$ vs the position $\bar{a}$ of a concentrated mass $\bar{M}_2$ ($\bar{M}_2 = \bar{J}_2 = 0.5$) of a fully fixed cantilever.

is more pronounced implying an increase in the flutter load by about 10.6%. This means that the presence of these parameters decreases the jump of the flutter load.

Finally, case 4 is associated with all characteristic response stages of the first case which are as indicated previously typical for all cases. In this case, as shown from Fig. 7 a tremendous jump in the flutter load is observed implying an increase in this load by about 300%. Comparing this jump with those corresponding to the previous cases it is clear that the elasticity of the support decreases the magnitude of discontinuity in the flutter load.

It should be mentioned that Figs. 6 and 7 appear also in reference[6] but without the flutter jump details.

Note also that a considerable jump has been observed in the flutter load function vs the magnitude of a concentrated mass $\bar{M}$ placed at the support of a Beck's column with $\bar{C}_T = 1$ and $\bar{C}_R \to \infty$ [6]. If the transverse shear effect is included in the analysis the aforegoing jump decreases appreciably.

In view of the above development it is worth noticing that the discontinuity in the flutter load is always associated with the derivation of the flutter load by the coincidence of the second and third eigenfrequencies. Moreover, this discontinuity is a result of the combined effect of at least three of the parameters: $\bar{C}_R$, $\bar{C}_T$, $\bar{M}_i$, $\bar{J}_i$, $\bar{a}$.

Finally, it should be mentioned that the change in flutter mode due to the change in some mass parameter is not uncommon on flutter instability problems. For instance in Ref. [9] the change in mass ratio resulted in sudden jumps in the critical speed of flow. In this reference it was also shown that damping may play a significant role on the critical load.

CONCLUSIONS

From this investigation one can draw the following important conclusions:

1. A thorough study of cases where the flutter load is obtained through the coincidence of the second and third eigenfrequencies is presented.

2. This occurs as a result of the coupling effect of at least three parameters.

3. In all these cases a discontinuity with a finite jump in the flutter load occurs for a certain combination of the values of the parameters.

4. In every discontinuity the lower value of the flutter load function is obtained through the coincidence of the first and second eigenfrequencies, while the higher value of this load is obtained through the coincidence of the second and third eigenfrequencies.

5. From all cases considered, it is clear that the flutter load is a sectionally continuous function.

6. A little before this discontinuity there exists a small post-critical stable region between the first (obtained through the coincidence of the two first eigenfrequencies) and second (obtained through the coincidence of the second and third eigenfrequencies) flutter load, due to the reappearence of the two first eigenfrequencies.

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APPENDIX

The expressions of the functions $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\varphi_3(\xi)$, $\varphi_4(\xi)$ are given as below

$$\varphi_1(\xi) = F_1(\xi) - \left(k_1^4 + \frac{\bar{C}_T\bar{C}_R}{sr^2}\right)F_2'(\xi) + \left(k_1^4\lambda^{-2} - \frac{1}{s}\right)\bar{C}_T F_2(\xi)$$

$$\varphi_2(\xi) = F_2''(\xi) + \bar{C}_R F_2'(\xi)$$

$$\varphi_3(\xi) = \left[k_2^4F_2(\xi) - \frac{\bar{C}_T\bar{C}_R}{r^2}(k_1^4F_2(\xi) + F_2''(\xi)) - \bar{C}_T\left(\left(\frac{1}{s} + \frac{k_1^4}{r^2}\right)F_2'(\xi) + \frac{1}{r^2}F_2''(\xi)\right)\right]$$

$$\varphi_4(\xi) = [F_2'(\xi) + k_1^4F_2'(\xi) + \bar{C}_R[F_2''(\xi) + k_1^4F_2(\xi)]]s$$

in which

$$F_1(\xi) = \frac{\epsilon^2\cosh\zeta\xi + \zeta^2\cos\epsilon\xi}{\epsilon^2 + \zeta^2}; \qquad F_2(\xi) = \frac{\epsilon\sinh\zeta\xi - \zeta\sin\epsilon\xi}{\zeta\epsilon(\epsilon^2 + \zeta^2)}$$

$$\epsilon^2 = \frac{\beta^2}{2} + \sqrt{\frac{\beta^4}{4} + k^4}; \qquad \zeta^2 = -\frac{\beta^2}{2} + \sqrt{\frac{\beta^4}{4} + k^4}$$

$$\beta^2 = \Omega^2\left(\frac{1}{sr^2} + \frac{1}{\lambda^2}\right) + \frac{\bar{s}}{s}; \qquad k^4 = -\frac{\Omega^2}{s}\left(\frac{\Omega^2}{2\lambda^4k'(\nu+1)} - 1\right).$$